International Journal of Theoretical Physics, Vol. 35, No. 7, 1996

Gauge Invariance and Formal Integrability of the Yang–Mills–Higgs Equations

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Received July 28, 1995

In the framework of the formal theory of overdetermined systems of partial differential equations, it is shown that the Yang–Mills–Higgs equations are an involutive, and hence formally integrable, system. To this end a key role is played by the gauge invariance of the theory and the resulting differential identities involving the field equations themselves. By applying a theorem of Malgrange, an existence theorem for the solutions of the Yang–Mills–Higgs field equations in the analytic context is thus obtained. The approach is within differential geometry.

1. INTRODUCTION

In the last decades classical gauge theories have been widely studied by both mathematicians and physicists (Daniel and Viallet, 1980; Drechsler and Mayer, 1977; Giachetta and Mangiarotti, 1989, 1990; Mangiarotti and Modugno, 1985). In particular, the pure Yang–Mills and the Yang–Mills–Higgs equations have been analyzed, and in some cases solved, using a combination of analytic and geometric methods (Atiyah *et al.*, 1978; Atiyah and Hitchin, 1988; Garland and Murray, 1989; Jaffe and Taubes, 1980). The classical solutions of these equations (instantons, vortices, monopoles, etc.) have been revealed to be important not only on their own, but also for a partial understanding of quantum theory (Abers and Lee, 1973; Faddeev and Slavnov, 1991).

The aim of this paper is to study the geometric structure of the Yang–Mills–Higgs equations and their solutions. We do this by considering the formal theory of overdetermined systems of partial differential equations (pde's), particularly in the form given by Kuranishi (1967) and Goldschmidt

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0020-7748/96/0700-1405$09.50/0

(1967a,b). We show that the Yang–Mills–Higgs equations are an involutive system of pde's. Moreover, they are formally integrable. As a result we also prove an existence theorem for the solution of these equations in an analytic context.

Our work follows the lines of Gasqui (1982); DeTurck (1981), and Giachetta and Mangiarotti (1994), where other equations of physical interest have been studied within the formal theory of pde's.

The organization of the paper is as follows. In Section 2 we briefly summarize the main concepts of the jet bundle formalism. We also recall two important results: a theorem due to Goldschmidt (1967b), which provides sufficient conditions for the formal integrability, and a theorem by Malgrange (1972a,b), which in the analytic context guarantees the convergence of power-series solutions for pde's which are formally integrable.

In Section 3 we turn our attention to the Yang–Mills–Higgs equations following the lines outlined in the previous section. To the end of proving the formal integrability of these equations, a key role is played by the differential identities which involve the field equations themselves and result from the gauge-invariance property of the theory. Moreover, other simple geometric properties of the spaces and maps involved are used.

We hope that this paper may contribute to arousing interest in the formal theory of pde's and its use, which in our opinion is still widely unknown to mathematicians and physicists.

2. FORMAL THEORY OF PDE'S

2.1. Jet Formalism

Throughout the paper all manifolds and maps will be smooth (C^∞). Manifolds will always be paracompact, Hausdorff topological spaces. A standard reference for material on the jet formalism is Saunders (1989).

If M is a differentiable manifold, we denote by TM and T^*M its tangent and cotangent bundles, respectively. The pth tensor, symmetric, and exterior products of T^*M are denoted by $\otimes^p T^*M$, $\vee_p T^*M$, and $\wedge^p T^*M$, respectively. We identify $\vee_p T^*M$ and $\wedge^p T^*M$ with subbundles of $\otimes^p T^*M$ by defining

$$\alpha_1 \vee \cdots \vee \alpha_p = \sum_{\sigma \in S_p} \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(p)}$$

and

$$\alpha_1 \wedge \cdots \wedge \alpha_p = \sum_{\sigma \in S_p} \operatorname{sgn} \sigma \, \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(p)}$$

for all $\alpha_1, \ldots, \alpha_p \in T^*M$, where S_p is the group of permutations of $\{1, \ldots, p\}$ and sgn σ is the sign of $\sigma \in S_p$.

If $\varphi \in \wedge^p T^*M$ and $\psi \in \wedge^q T^*M$, we define the wedge product $\varphi \wedge \psi \in \wedge^{p+q} T^*M$ by

$$\varphi \wedge \psi(X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sgn} \sigma \, \varphi(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \psi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})$$

for all $X_1, \ldots, X_{p+q} \in TM$.

Let $\pi: E \to M$ be a fibered manifold (i.e., a surjective submersion), with

$$m = \dim M \qquad \text{and} \qquad m + l = \dim E$$

The standard chart of E is denoted by (x^λ, y^i), with $1 \leq \lambda \leq m$ and $1 \leq i \leq l$.

Let J^kE be the kth-order jet prolongation of π, with $k \geq 0$ ($J^0E = E$). This is naturally a fibered manifold $\pi^k: J^kE \to M$ and a fiber bundle $\pi^k_h: J^kE \to J^hE$, respectively, with $0 \leq h \leq k$. The standard chart of J^kE is denoted by (x^λ, y^i_B), with $0 \leq |B| \leq k$, where $B = (B_1, \ldots, B_m)$ is a multi-index and $|B| = B_1 + \cdots + B_m$. We put

$$O = (0, \ldots, 0), \qquad B + \lambda = (B_1, \ldots, B_\lambda + 1, \ldots, B_m) \tag{2.1}$$
$$y^i = y^i_O, \qquad y^i_\lambda = y^i_{O+\lambda}, \qquad y^i_{\lambda\mu} = y^i_{O+\lambda+\mu}, \ldots$$

If $s: M \to E$ is a (local) section, then $j^ks: M \to J^kE$ is its kth-order jet prolongation, whose coordinate expression is

$$(x^\lambda, y^i_B) \circ j^ks = (x^\lambda, \partial_B s^i)$$

where

$$s^i = y^i \circ s \qquad \text{and} \qquad \partial_B s^i = \partial_1^{B_1} \cdots \partial_m^{B_m} s^i$$

We will always put $j^k_x s \equiv j^ks(x)$ for any x in the neighborhood where s is defined.

We have a basic affine structure on jet manifolds; namely $\pi^k_{k-1}: J^kE \to J^{k-1}E$ is an affine bundle, for $k \geq 1$, whose vector bundle is the pullback bundle

$$J^{k-1}E \underset{E}{\times} \vee_k T^*M \otimes VE$$

As is well known, if $\pi: E \to M$ is a vector bundle, then $\pi^k: J^kE \to M$ is a vector bundle, too. Moreover, there is a morphism of vector bundles (over M)

$$\epsilon: \quad \vee_k T^*M \otimes E \to J^kE$$

given by

$$\epsilon: \quad u^i_B \mapsto \begin{cases} y^i_B = 0 & \text{if} \quad 0 \leq |B| \leq k-1 \\ y^i_B = u^i_B & \text{if} \quad |B| = k \end{cases} \tag{2.2}$$

where (x^λ, u^i_B) is the standard chart of $\vee_k T^*M \otimes E$.

A pde of order k on E is defined to be a fibered submanifold $R^k \to M$ of π^k: $J^kE \to M$. A solution of R^k is a (local) section s: $M \to E$ such that its kth-order jet prolongation j^ks is a section of π^k: $R^k \to M$.

Let π': $E' \to M$ be another fibered manifold and let Φ: $J^kE \to E'$ be a morphism of fibered manifolds over M. Given a section s': $M \to E'$, we define

$$R^k = \text{Ker}_{s'}\Phi = \{p \in J^kE\colon \Phi(p) = s' \circ \pi^k(p)\} \tag{2.3}$$

and assume that

$$s'(M) \subset \text{Im}(\Phi) \qquad \text{and} \qquad \Phi \text{ has locally constant rank} \tag{2.4}$$

Then one can show that R^k is pde of order k. Note that any pde R^k of order k on E can be locally described as in (2.3) and (2.4).

Let (x^λ, y'^r) be the standard chart of E', with $1 \leq r \leq l'$. Then locally the morphism Φ is determined by the l' functions

$$\Phi^r(x^\lambda, y^i_B), \qquad 0 \leq |B| \leq k$$

equal to $y'^r \circ \Phi$ and the pde R^k in (2.3) is defined by the equations

$$\Phi^r(x^\lambda, y^i_B) = s'^r(x^\lambda), \qquad 0 \leq |B| \leq k$$

In what follows we shall be interested in a distinguished class of pde's. Let π': $E' \to M$ be a vector bundle and let Φ: $J^kE \to E'$ be a morphism of fibered manifolds. We say that Φ is *quasilinear* if there exists a morphism of vector bundles

$$\sigma(\Phi)\colon \quad \vee_k T^*M \otimes VE \to E' \tag{2.5}$$

over $J^{k-1}E$ such that

$$\Phi(p + (\pi^k_{k-1}(p), u)) = \Phi(p) + \sigma(\Phi)(\pi^k_{k-1}(p), u)$$

for every $p \in J^kE$ and $u \in \vee_k T^*M \otimes VE$ projecting over the same point of E. In other words, Φ is quasilinear if it is an affine morphism with respect to the affine structure of J^kE over $J^{k-1}E$. The map $\sigma(\Phi)$, uniquely determined by Φ, is called the *symbol* of Φ and the pde defined in (2.3), (2.4) is said to be a *quasilinear* pde.

In the standard charts a quasilinear morphism Φ is determined by the functions

$$\Phi^r(x^\lambda, y^i_B, y^i_C) = \sum_{|C|=k} \psi^{rC}_j(x^\lambda, y^i_B) y^j_C + \psi^r(x^\lambda, y^i_B),$$

$$0 \leq |B| \leq k - 1, \quad |C| = k$$

with $\psi^{rC}_j(x^\lambda, y^i_B)$ defining the symbol $\sigma(\Phi)$ of Φ.

As is easily seen, if Φ is quasilinear and $\sigma(\Phi)$ is an epimorphism, then Φ is a surjective submersion. Hence (2.4) is satisfied and

$$\pi^k_{k-1}(R^k) = J^{k-1}E \tag{2.6}$$

Usually one denotes by

$$G^k \subset R^k \underset{E}{\times} \vee_k T^*M \otimes VE$$

the pullback over R^k of the kernel of $\sigma(\Phi)$. Sometimes G^k itself is called the *symbol* of Φ. Of course, it has a structure of vector bundle over R^k only under suitable regularity conditions of $\sigma(\Phi)$.

We now introduce a basic concept of the theory of pde's, namely that of prolongation. Let Φ: $J^kE \to E'$ and s': $M \to E'$ be as in (2.4). The hth-order prolongation of Φ ($h \geq 0$) is the morphism

$$p_h(\Phi): \quad J^{k+h}E \to J^hE'$$

(over M) defined by

$$p_h(\Phi)(j^{k+h}_x s) = j^h_x(\Phi \circ j^k s) \tag{2.7}$$

for every $x \in M$ and section s: $M \to E$ [$p_0(\Phi) = \Phi$]. The subset

$$R^{k+h} = \mathrm{Ker}_{j^h s'}\, p_h(\Phi) \subset J^{k+h}E \tag{2.8}$$

is called the hth-order *prolongation* of the pde R^k. For example, the first-order prolongation $p_1(\Phi)$ of Φ is determined in the standard charts by

$$\Phi^r(x^\lambda, y^i_B)$$
$$\partial_\mu \Phi^r(x^\lambda, y^i_B) + \partial^C_j \Phi^r(x^\lambda, y^i_B) y^j_{C+\mu}$$

It can be shown (Pommaret, 1978) that the prolongation depends on R^k only, and not on the morphism Φ defining it. Obviously, the canonical projections π^{k+h+1}_{k+h}: $J^{k+h+1}E \to J^{k+h}E$, $h \geq 0$, restrict to maps π^{k+h+1}_{k+h}: $R^{k+h+1} \to R^{k+h}$.

One can easily check that if Φ is quasilinear, then so is its prolongation of any order. The uniquely determined morphism of vector bundles

$$\sigma_h(\Phi): \quad \vee_{k+h} T^*M \otimes VE \to \vee_h T^*M \otimes E' \tag{2.9}$$

(over $J^{k-1}E$) is called the hth-order *prolongation* of the symbol $\sigma(\Phi)$. It is given by the composition

$$\vee_{k+h}T^*M \otimes VE \overset{i}{\hookrightarrow} \vee_h T^*M \otimes \vee_k T^*M \otimes VE \overset{\sigma(\Phi)}{\longrightarrow} \vee_h T^*M \otimes E' \tag{2.10}$$

where i denotes the canonical inclusion.

The pullback over R^k of the kernel of $\sigma_h(\Phi)$ is denoted by

$$G^{k+h} \subset R^k \underset{E}{\times} \vee_{k+h}T^*M \otimes VE$$

This may fail to be a vector bundle even if G^k is a vector bundle.

2.2. A Criterion for Formal Integrability

If M is a real-analytic manifold, E and E' are real-analytic fibered manifolds over M, Φ: $J^kE \to E'$ is a real-analytic morphism, and s': $M \to E'$ is a real-analytic section, then the pde defined in (2.3), (2.4) is said to be *analytic*.

Given an analytic pde R^k of order k, we are interested in finding its convergent power-series solutions in a neighborhood of any point $x \in M$. We call a point of R^{k+h}, $h \geq 0$, a formal solution of R^k of order $k + h$ and a point of $R^\infty = \text{proj lim } R^{k+h}$ a formal solution. Of course, the construction of analytic solutions of R^k demands a preliminary step. This consists in seeking whether a formal solution of any order $\geq k$ can be prolonged to a formal solution. A sufficient condition is obviously that

$$\text{the maps } \pi^{k+h+1}_{k+h}\colon \quad R^{k+h+1} \to R^{k+h},\ h \geq 0, \text{ are surjective} \tag{2.11}$$

Then the following important theorem (Malgrange, 1972a,b) guarantees the existence of convergent power-series solutions for analytic pde's satisfying (2.11).

Theorem 2.1. Let R^k be an analytic pde defined as in (2.3). Let $x \in M$ and $h \geq 0$. If π^{k+n+1}_{k+n}: $R^{k+n+1}_x \to R^{k+n}_x$ is surjective for all $n \geq h$, then for every point $p \in R^{k+h}_x$ there exists an analytic solution s: $U \subset M \to E$ of R^k over a neighborhood U of x such that $j^{k+h}_x s = p$.

In general a direct check of (2.11) is not simple. Nevertheless, there are criteria which allow us to verify the surjectivity of all maps (2.11) in a finite number of steps. We say that R^k is *formally integrable* if all maps (2.11) are surjective submersions. The following theorem is due to Goldschmidt (1967b) and gives us sufficient conditions for formal integrability. Combined with Theorem 2.1, it leads to the existence of analytic solutions of analytic quasilinear pde's.

Theorem 2.2. Let Φ: $J^kE \to E'$ be a quasilinear morphism and let R^k be the corresponding quasilinear pde defined in (2.3) and (2.4). If

(i) G^{k+1} is a vector bundle over R^k
(ii) π_k^{k+1}: $R^{k+1} \to R^k$ is surjective
(iii) G^k is 2-acyclic

then R^k is formally integrable.

The condition (iii) refers to the vanishing of some of the Spencer cohomology groups $H^{k-j,j}(G^k)$. We do not need to go into details here for these cohomology groups because we replace (iii) with a stronger condition, namely:

(iii′) For all $p \in R^k$ there exists a quasiregular basis of $T_{\pi^k(p)}M$ for G^k at p.

This means the following. Let $x \in M$ and let (X_λ), with $1 \leq \lambda \leq m$, be a basis of T_xM. If (θ^λ) is the basis of T^*M dual to (X_λ), then we denote by $\vee_{k,j} T_x^*M$ the subspace of $\vee_k T_x^*M$ spanned by $\theta^{\mu_1} \vee \cdots \vee \theta^{\mu_k}$, with $j + 1 \leq \mu_1 \leq \cdots \leq \mu_k \leq m$. For every $p \in R^k$ we define

$$(G^k)_{p,j} = (G^k)_p \cap \vee_{k,j}T_x^*M \otimes (VE)_e$$

where $x = \pi^k(p)$ and $e = \pi_0^k(p)$. One says that (X_λ) is a *quasiregular* basis for G^k at p if

$$\dim(G^{k+1})_p = \dim(G^k)_p + \sum_{j=1}^{m-1} \dim(G^k)_{p,j} \qquad (2.12)$$

The condition (iii′) corresponds to the *involutivity* of the symbol G^k of R^k. A pde R^k is said to be *involutive* if it is formally integrable and its symbol G^k is involutive.

3. YANG–MILLS–HIGGS EQUATIONS

3.1. Geometric Setup of the Yang–Mills–Higgs pde's

Let $P \to M$ be a principal fiber bundle (dim $M > 1$) with a compact structure Lie group G (Kobayashi and Nomizu, 1963). We denote by $C \to M$ the bundle of all principal connections on P. As is well known, this is an affine bundle whose associated vector bundle is $T^*M \otimes V_GP \to M$. Here the quotient bundle $V_GP = VP/G \to M$ is the vector bundle of right invariant vertical vector fields on P. The standard charts of C, J^1C, and J^2C are denoted by (x^λ, a_λ^r), $(x^\lambda, a_\lambda^r, a_{\mu,\lambda}^r)$, and $(x^\lambda, a_\lambda^r, a_{\mu,\lambda}^r, a_{\nu\mu,\lambda}^r)$, respectively [see (2.1)].

Let A: $M \to C$ be a section, i.e., a principal connection on P (gauge potential). Locally we write

$$(x^\lambda, a^r_\lambda) \circ A = (x^\lambda, A^r_\lambda)$$

where A^r_λ are local functions on M. The curvature of A (field strength) is the following V_GP-valued 2-form on M:

$$F_A: \quad M \to \overset{2}{\wedge} T^*M \otimes V_GP, \qquad F_A = \tfrac{1}{2} F^r_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes e_r \tag{3.1}$$

$$F^r_{\lambda\mu} = \partial_\lambda A^r_\mu - \partial_\mu A^r_\lambda + c^r_{pq} A^p_\lambda A^q_\mu$$

(where e_r) is a basis of the Lie algebra of G and the Lie bracket $[e_p, e_q] = c^r_{pq} e_r$ defines the right structure constants of G.

Let V be a vector space on which G acts as a transformation group and let $E \to M$ be the corresponding associated vector bundle. Sections ϕ: $M \to E$ of this bundle are scalar (Higgs) fields. Standard coordinates on E, J^1E, and J^2E are denoted by (x^λ, φ^i), $(x^\lambda, \varphi^i, \varphi_{\mu,}{}^i)$, and $(x^\lambda, \varphi^i, \varphi_{\mu,}{}^i, \varphi_{\nu\mu,}{}^i)$, respectively. Locally we write

$$(x^\lambda, \varphi^i) \circ \phi = (x^\lambda, \phi^i)$$

where ϕ^i are local functions on M. Let ρ: $LG \to \mathrm{End}(V)$ be the Lie algebra representation induced by the action of G on V.

A principal connection A induces linear connections on the vector bundles $V_GP \to M$ and $E \to M$. We denote both of them by the same symbol ∇^A. Their connection parameters are determined by the following equations:

$$\nabla^A e_q = c^r_{pq} A^p_\lambda dx^\lambda \otimes e_r \tag{3.2}$$

$$\nabla^A e_j = -\rho^i_{pj} A^p_\lambda dx^\lambda \otimes e_i \tag{3.3}$$

respectively. Here (e_i) is a basis of V and $\rho^i_{pj} = \langle e^i, \rho(e_r)e_j \rangle$. The Higgs field couples 'minimally' to the connection through the covariant derivative

$$\gamma_{A,\phi} \equiv \nabla^A\phi: M \to T^*M \otimes E, \qquad \gamma_{A,\phi} = \gamma^i_\lambda dx^\lambda \otimes e_i \tag{3.4}$$

$$\gamma^i_\lambda = \partial_\lambda \phi^i - \rho^i_{pj} A^p_\lambda \phi^j$$

Assume that M is an oriented manifold which carries a metric g of any signature. Let h and k be inner products on LG and V, respectively, such that the adjoint representation and the representation of G on V are unitary. We want to study the geometric structure of the following system of pde's:

$$\nabla^A * F_A = J_{A,\phi} \tag{3.5a}$$

$$\nabla^A * \gamma_{A,\phi} = 0 \tag{3.5b}$$

where $*$ is the Hodge operator and $J_{A,\phi}: M \to \wedge^{m-1}T^*M \otimes V_G^*P$ is the current. These are the Yang–Mills–Higgs equations with vanishing self-interaction term. In coordinates they read

$$\partial_\lambda(\sqrt{|g|}F_r^{\lambda\mu}) - \sqrt{|g|}c_{qr}^s A_\lambda^q F_s^{\lambda\mu} + \sqrt{|g|}\rho_{rj}^i \phi^j \gamma_i^\mu = 0 \tag{3.6a}$$

$$\partial_\lambda(\sqrt{|g|}\gamma_j^\lambda) + \sqrt{|g|}\rho_{rj}^i A_\lambda^r \gamma_i^\lambda = 0 \tag{3.6b}$$

Here we have defined

$$F_r^{\lambda\mu} = h_{rs} g^{\lambda\alpha} g^{\mu\beta} F_{\alpha\beta}^s \tag{3.7}$$

$$\gamma_i^\lambda = k_{ij} g^{\lambda\mu} \gamma_\mu^j \tag{3.8}$$

and $g = \det(g_{\lambda\mu})$, where $g_{\lambda\mu}$, h_{rs}, and k_{ij} are the coordinate expressions of the metrics g, h, and k, respectively. The current $J_{A,\phi}$ is given by

$$J_{A,\phi} = -\rho^* * \gamma_{A,\phi} \tag{3.9}$$

$$J_{A,\phi} = -\sqrt{|g|}\rho_{rj}^i \phi^j \gamma_i^\lambda \omega_\lambda \otimes e^r$$

with

$$\omega_\lambda = \partial_\lambda \rfloor \omega \qquad \text{and} \qquad \omega = dx^1 \wedge \cdots \wedge dx^m$$

ρ^* is a sort of pullback bringing E^*-valued forms on M to V_G^*P-valued forms and whose definition is shown in the local expression (3.9).

Let $L = C \oplus E$ be the Whitney sum of the bundles $C \to M$ and $E \to M$. We call L the Yang–Mills–Higgs configuration bundle. Let us consider the morphism

$$\Phi: \quad J^2L \to \overset{m-1}{\wedge} T^*M \otimes V_G^*P \oplus \overset{m}{\wedge} T^*M \otimes E^* \tag{3.10}$$

$$\Phi(j_x^2 A, j_x^2 \phi) = [\nabla^A * F_A(x) - J_{A,\phi}(x), \nabla^A * \gamma_{A,\phi}(x)]$$

for all $x \in M$ and sections $(A, \phi): M \to L$. From (3.7)–(3.9), (3.1), and (3.4) it is easily seen that Φ is quasilinear. According to (2.3), we define

$$R^2 = \mathrm{Ker}_{\bar{0}}\Phi \equiv \mathrm{Ker}\Phi \subset J^2L \tag{3.11}$$

where $\bar{0}$ is the zero section of

$$\overset{m-1}{\wedge} T^*M \otimes V_G^*P \oplus \overset{m}{\wedge} T^*M \otimes E^* \to M$$

A pair (A, ϕ) formed by a principal connection $A: M \to C$ and a Higgs field $\phi: M \to E$ is a solution of the Yang–Mills–Higgs equations if and only if its second-order prolongation $(j^2A, j^2\phi): M \to J^2L$ takes values in R^2.

3.2. Involutivity of the Yang–Mills–Higgs pde's

In what follows we will show that (3.11) is an involutive pde. To begin with, let us verify that R^2 is a fibered submanifold of $J^2L \to M$, that is, a pde of second order. To this end consider the symbol

$$\sigma(\Phi):\quad \vee_2 T^*M \otimes \overline{L} \to \overset{m-1}{\wedge} T^*M \otimes V_G^*P \oplus \overset{m}{\wedge} T^*M \otimes E^* \tag{3.12}$$

of Φ. Here

$$\overline{L} = T^*M \otimes V_G P \oplus E \to M$$

is the vector bundle on which $L \to M$ is modeled. Denoting the standard chart of $\vee_2 T^*M \otimes \overline{L}$ by $(x^\lambda, u_{\nu\mu\lambda}{}^r, v_{\nu\mu}{}^i)$, from (3.7)–(3.9), (3.1), and(3.4) we find that

$$\begin{aligned} \sigma(\Phi):&\quad u_{\nu\mu\lambda}{}^r \mapsto \sqrt{|g|}\,h_{rs} g^{\lambda\alpha} g^{\mu\beta}(u_{\lambda\alpha\beta}{}^s - u_{\lambda\beta\alpha}{}^s) \\ \sigma(\Phi):&\quad v_{\nu\mu}{}^i \mapsto \sqrt{|g|}\,k_{ij} g^{\lambda\mu} v_{\lambda\mu}{}^j \end{aligned} \tag{3.13}$$

Note that the symbol $\sigma(\Phi)$ is constant along the fibers of $J^1L \to M$. Note also that it is the direct sum of two symbols, namely

$$\sigma(\Phi) = \sigma(\nabla * F) \oplus \sigma(\nabla * \gamma) \tag{3.14}$$

with

$$\sigma(\nabla * F):\quad \vee_2 T^*M \otimes T^*M \otimes V_G P \to \overset{m-1}{\wedge} T^*M \otimes V_G^*P$$

and

$$\sigma(\nabla * \gamma):\quad \vee_2 T^*M \otimes E \to \overset{m}{\wedge} T^*M \otimes E^*$$

Here $\nabla * F$ and $\nabla * \gamma$ denote the Yang–Mills and Higgs operators, respectively.

One can easily check that $\sigma(\nabla * F)$ is determined by the following composition of morphisms:

$$\vee_2 T^*M \otimes T^*M \otimes V_G P \xrightarrow{\xi} T^*M \otimes V_G P \xrightarrow{\eta} \overset{m-1}{\wedge} T^*M \otimes V_G^*P \tag{3.15}$$

where ξ acts as the identity on $V_G P$, while

$$\xi(\alpha \vee \beta \otimes \gamma) = 2g(\alpha, \beta)\gamma - g(\alpha, \gamma)\beta - g(\beta, \gamma)\alpha \tag{3.16}$$

for every g: $M \to L$, α, β, $\gamma \in T^*M$; η is the tensor product morphism of the Hodge operator on M and the metric isomorphism on $V_G P$ induced by h. The other symbol $\sigma(\nabla * \gamma)$ is the tensor product of the metric isomorphism between E and E^* induced by k and the morphism

$$\zeta:\quad \vee_2 T^*M \to \overset{m}{\wedge} T^*M, \qquad \zeta:\quad \alpha \vee \beta \mapsto 2g(\alpha, \beta)\sqrt{|g|}\,\omega \tag{3.17}$$

for all $\alpha, \beta \in T^*M$.

Lemma 3.1. The symbol $\sigma(\Phi)$ is a surjective morphism.

Proof. We show that both $\sigma(\nabla * F)$ and $\sigma(\nabla * \gamma)$ are surjective morphisms. For every $x \in M$, let (dx^λ) be an orthonormal basis of $T_x^* M$, i.e.,

$$g(dx^\lambda, dx^\mu) = \begin{cases} 0 & \text{if} \quad \lambda \neq \mu \\ \pm 1 & \text{if} \quad \lambda = \mu \end{cases}$$

Then consider the equations

$$2g(\alpha, \beta)\gamma - g(\alpha, \gamma)\,\beta - g(\beta, \gamma)\alpha = dx^\lambda, \qquad 1 \leq \lambda \leq m$$

Solutions to these equations are easily found; for example, $\gamma = g^{\mu\mu} dx^\lambda$ and $\alpha = \beta = 1/\sqrt{2}\; dx^\mu$, with $\mu \neq \lambda$. Hence ξ is surjective. Since η is an isomorphism, it follows that $\sigma(\nabla * F)$ is surjective. The surjectivity of $\sigma(\nabla * \gamma)$ is evident. ■

According to (2.6) and the comment preceding it, an immediate consequence of this lemma is that $R^2 \subset J^2L$ is a fibered submanifold over M and

$$\pi_1^2(R^2) = J^1L \tag{3.18}$$

Now we consider the first-order prolongation of Φ, i.e.,

$$p_1(\Phi):\quad J^3L \to J^1(\overset{m-1}{\wedge} T^*M \otimes V_G^*P \oplus \overset{m}{\wedge} T^*M \otimes E^*) \tag{3.19}$$

From (2.7) and (3.10) we see that

$$p_1(\Phi)(j_x^3 A, j_x^3 \phi) = [j_x^1(\nabla^A * F_A - J_{A,\phi}), j_x^1(\nabla^A * \gamma_{A,\phi})] \tag{3.20}$$

for all $x \in M$ and sections (A, ϕ): $M \to L$. Let

$$\sigma_1(\Phi):\quad \vee_3 T^*M \otimes \overline{L} \to T^*M \otimes (\overset{m-1}{\wedge} T^*M \otimes V_G^*P \oplus \overset{m}{\wedge} T^*M \otimes E^*)$$

be the first-order prolongation of the symbol (3.12). Its coordinate expression can be obtained directly from (2.10) and (3.13),

$$\begin{aligned} \sigma_1(\Phi):&\quad u_{\alpha\nu\mu\lambda}{}^r \mapsto \sqrt{|g|}\,h_{rs} g^{\lambda\alpha} g^{\mu\beta}(u_{\nu\lambda\alpha\beta}{}^s - u_{\nu\lambda\beta\alpha}{}^s) \\ \sigma_1(\Phi):&\quad v_{\alpha\nu\mu}{}^i \mapsto \sqrt{|g|}\,k_{ij} g^{\lambda\mu} v_{\nu\lambda\mu}{}^j \end{aligned} \tag{3.21}$$

where $(x^\lambda, u_{\alpha\nu\mu\lambda}{}^r, v_{\alpha\nu\mu}{}^i)$ is the standard chart of $\vee_3 T^*M \otimes \overline{L}$. Moreover, from (2.10) and (3.14) we see that

$$\sigma_1(\Phi) = \sigma_1(\nabla * F) \oplus \sigma_1(\nabla * \gamma) \tag{3.22}$$

where

$$\sigma_1(\nabla * F):\quad \vee_3 T^*M \otimes T^*M \otimes V_G P \to T^*M \otimes \overset{m-1}{\wedge} T^*M \otimes V_G^* P$$

and

$$\sigma_1(\nabla * \gamma):\quad \vee_3 T^*M \otimes E \to T^*M \otimes \overset{m}{\wedge} T^*M \otimes E^*$$

are the first-order prolongations of the symbols of the Yang–Mills and Higgs operators, respectively. In accordance with Section 2.1, we denote the kernel of $\sigma_1(\Phi)$ by G^3.

Next we show that the conditions (i) and (ii) of Theorem 2.2 hold. Let A: $M \to C$ be a principal connection and let

$$\Psi_A:\quad J^1(\overset{m-1}{\wedge} T^*M \otimes V_G^* P) \to \overset{m}{\wedge} T^*M \otimes V_G^* P$$

be the morphism (over M) corresponding to the operator of covariant differentiation with respect to A, i.e.,

$$\Psi_A(j_x^1\theta) = (\nabla^A\theta)(x) \tag{3.23}$$

for all $x \in M$ and sections θ: $M \to \wedge^{m-1}T^*M \otimes V_G^*P$. Recalling (3.2), we have

$$\Psi_A:\quad (x^\lambda, \theta_r^\lambda, \partial_\mu\theta_r^\lambda) \mapsto (x^\lambda, \partial_\lambda\theta_r^\lambda - c_{qr}^s A_\lambda^q \theta_s^\lambda)$$

Note that Ψ_A is a linear morphism. Its symbol

$$\sigma(\Psi):\quad T^*M \otimes \overset{m-1}{\wedge} T^*M \otimes V_G^* P \to \overset{m}{\wedge} T^*M \otimes V_G^* P$$

is simply given by the wedge product. Note also that $\sigma(\Psi)$ actually does not depend on the connection A. For the sake of brevity, let us denote by the same symbol $\sigma(\Psi)$ the morphism

$$\sigma(\Psi):\quad T^*M \otimes (\overset{m-1}{\wedge} T^*M \otimes V_G^* P \oplus \overset{m}{\wedge} T^*M \otimes E^*) \to \overset{m}{\wedge} T^*M \otimes V_G^* P$$

defined as the composition of $\sigma(\Psi)$ with the natural projection onto the first factor, i.e.,

$$\begin{aligned} &T^*M \otimes (\overset{m-1}{\wedge} T^*M \otimes V_G^* P \oplus \overset{m}{\wedge} T^*M \otimes E^*) \\ &\to T^*M \otimes \overset{m-1}{\wedge} T^*M \otimes V_G^* P \xrightarrow{\sigma(\Psi)} \overset{m}{\wedge} T^*M \otimes V_G^* P \end{aligned}$$

The following lemma tells us that G^3 is a vector bundle.

Lemma 3.2. The sequence

$$0 \to G^3 \to \vee_3 T^*M \otimes \bar{L} \xrightarrow{\sigma_1(\Phi)} T^*M \otimes (\overset{m-1}{\wedge} T^*M \otimes V_G^*P \oplus \overset{m}{\wedge} T^*M \otimes E^*)$$
$$\xrightarrow{\sigma(\Psi)} \overset{m}{\wedge} T^*M \otimes V_G^*P \to 0$$

is exact.

Proof. Notice that the sequence decomposes into two other sequences,

$$0 \to (G_1)^3 \to \vee_3 T^*M \otimes T^*M \otimes V_G P \xrightarrow{\sigma_1(\nabla * F)} T^*M \otimes \overset{m-1}{\wedge} T^*M \otimes V_G^*P$$
$$\xrightarrow{\sigma(\Psi)} \overset{m}{\wedge} T^*M \otimes V_G^*P \to 0$$

and

$$0 \to (G_2)^3 \to \vee_3 T^*M \otimes E \xrightarrow{\sigma_1(\nabla * \gamma)} T^*M \otimes \overset{m}{\wedge} T^*M \otimes E^* \to 0$$

We begin by proving that $\sigma_1(\nabla * \gamma)$ is surjective. By using (3.17) and neglecting E and E^* in the tensor products, we find that

$$\sigma_1(\nabla * \gamma)(\alpha \vee \beta \vee \delta) = 2[g(\alpha, \beta)\delta + g(\alpha, \delta)\beta + g(\beta, \delta)\alpha] \otimes \sqrt{|g|}\,\omega$$

for all α, β, and $\delta \in T^*M$. Let (dx^λ) be an orthonormal basis of T_x^*M, $x \in M$. Then the equations

$$\sigma_1(\nabla * \gamma)(\alpha \vee \beta \vee \delta) = dx^\lambda \otimes \omega, \qquad 1 \le \lambda \le m$$

are solved by $\alpha = \beta = dx^\mu$ and $\delta = \frac{1}{2}g^{\mu\mu}dx^\lambda$, $\mu \neq \lambda$. As for the other sequence, from (3.21) we find

$$\sigma(\Psi) \circ \sigma_1(\nabla * F): \quad u_{\alpha\nu\mu\lambda}{}^r \mapsto \sqrt{|g|}\,h_{rs} g^{\lambda\alpha} g^{\mu\beta}(u_{\mu\lambda\alpha\beta}{}^s - u_{\mu\lambda\beta\alpha}{}^s)$$
$$= \sqrt{|g|}\,h_{rs}(g^{\lambda\alpha} g^{\mu\beta} u_{\mu\lambda\alpha\beta}{}^s - g^{\mu\beta} g^{\lambda\alpha} u_{\lambda\mu\alpha\beta}{}^s) = 0$$

so that $\operatorname{Im} \sigma_1(\nabla * F) \subset \operatorname{Ker} \sigma(\Psi)$. Now we show that $\operatorname{Im} \sigma_1(\nabla * F) \supset \operatorname{Ker} \sigma(\Psi)$. Actually we prove the inclusion $\operatorname{Im}(\xi \circ i) \supset \operatorname{Ker}(\sigma(\Psi) \circ \eta)$, which is equivalent to the above one since $\eta: T^*M \otimes V_G P \to \wedge^{m-1} T^*M \otimes V_G^*P$ is an isomorphism. Recalling (3.16) and neglecting $V_G P$ and V_G^*P in the tensor products, we find that

$$\xi \circ i: \quad \vee_3 T^*M \otimes T^*M \to T^*M \otimes T^*M$$

is given by

$$\xi \circ i(\alpha \vee \beta \vee \gamma \otimes \delta) = 2g(\alpha, \beta)\gamma \otimes \delta + 2g(\alpha, \gamma)\beta \otimes \delta + 2g(\beta, \gamma)\alpha \otimes \delta$$
$$- g(\alpha, \delta)\beta \vee \gamma - g(\beta, \delta)\alpha \vee \gamma - g(\gamma, \delta)\alpha \vee \beta \tag{3.24}$$

for all $\alpha, \beta, \gamma, \delta \in T^*M$. In particular,

$$\xi \circ i(\alpha \vee \alpha \vee \alpha \otimes \delta) = 6\alpha \otimes [g(\alpha, \alpha)\delta - g(\alpha, \delta)\alpha] \tag{3.25}$$

On the other hand, as one can easily verify, $\mathrm{Ker}(\sigma(\Psi) \circ \eta)$ is generated by tensors of the type $\mu \otimes \mu$, with $g(\mu, \mu) = 0$, and $\mu \otimes \rho$, with $g(\mu, \rho) = 0$, $g(\mu, \mu) \neq 0$, $g(\rho, \rho) \neq 0$; $\mu, \rho \in T^*M$. Hence we are left with the problem of solving the equations

$$6\alpha \otimes [g(\alpha, \alpha)\delta - g(\alpha, \delta)\alpha] = \mu \otimes \mu$$

$$6\alpha \otimes [g(\alpha, \alpha)\delta - g(\alpha, \delta)\alpha] = \mu \otimes \rho$$

These are solved by

$$\alpha = \mu, \qquad \delta \text{ such that } \quad -6g(\mu, \delta) = 1$$

and

$$\alpha = \mu, \qquad \delta = \frac{1}{6g(\mu, \mu)}\rho$$

respectively. Obviously $\sigma(\Psi)$ is a surjective morphism. Therefore the lemma is proved. ∎

Our next task is to verify the condition (ii) of Theorem 2.2.

Lemma 3.3. The map π^3_2: $R^3 \to R^2$ is surjective.

Proof. Let $p \in R^2$ and let (A, ϕ): $M \to L$ be a section such that $p = (j^2_x A, j^2_x \phi)$. Consider then

$$\epsilon^{-1} \circ p_1(\Phi)(j^3_x A, j^3_x \phi) \in T^*_x M \otimes (\overset{m-1}{\wedge} T^*_x M \otimes (V^*_G P)_x \oplus \overset{m}{\wedge} T^*_x M \otimes E^*_x)$$

Since $p_1(\Phi)$ is quasilinear, the fiber $(R^3)_p$ is not empty iff $\epsilon^{-1} \circ p_1(\Phi)(j^3_x A, j^3_x \phi) \in \mathrm{Im}\, \sigma_1(\Phi)$ or equivalently, due to Lemma 3.2, iff $\sigma(\Psi) \circ \epsilon^{-1} \circ p_1(\Phi)(j^3_x A, j^3_x \phi) = 0$. From (3.20) and (3.23) and since Ψ_A is linear, we get

$$\begin{aligned} \sigma(\Psi) \circ \epsilon^{-1} \circ p_1(\Phi)(j^3_x A, j^3_x \phi) &= \Psi_A[j^1_x(\nabla^A * F_A - J_{A,\phi})] \\ &= (\nabla^A \nabla^A * F_A - \nabla^A J_{A,\phi})(x) \end{aligned}$$

Now using the differential identities

$$\nabla^A \nabla^A * F_A = 0, \qquad -\nabla^A J_{A,\phi} = \rho^* \nabla^A * \gamma_{A,\phi}$$

we find that

$$\sigma(\Psi) \circ \epsilon^{-1} \circ p_1(\Phi)(j^3_x A, j^3_x \phi) = (\rho^* \nabla^A * \gamma_{A,\phi})(x)$$

and the result follows by using the field equation (3.5b). The first of the above identities is a well-known identity involving the curvature of the connection, whereas the other is a consequence of the unitarity of the representation of G on V. ■

Remark 3.4. Let us consider the case where a Higgs self-interaction potential $U: E \to \wedge^m T^*M$ is present. Then the second Yang–Mills–Higgs equation (3.5b) becomes

$$\nabla^A * \gamma_{A,\phi} = f_\phi \tag{3.26}$$

where $f \equiv DU: E \to \wedge^m T^*M \otimes E^*$ denotes the fiber derivative. In this case the preceding discussion still holds true apart from Lemma 3.3. Indeed now we have

$$\sigma(\Psi) \circ \epsilon^{-1} \circ p_1(\Phi)(j_x^3 A, j_x^3 \phi) = (\rho^* f_\phi)(x)$$

and hence $R^3 \to R^2$ is surjective iff $\rho^* f: E \to \wedge^m T^*M \otimes V_G^* P$ vanishes identically.

Consider the Lie algebra representation

$$X: \quad M \to V_G P \mapsto u_X: \quad E \to VE$$

induced by $\rho: LG \to \mathrm{End}(V)$. Locally it reads (Giachetta and Mangiarotti, 1990)

$$X = X^r e_r \mapsto u_X = \rho^i_{rj} X^r \varphi^j \frac{\partial}{\partial \varphi^i}$$

As one can easily verify,

$$\mathcal{L}_{u_X} U = \langle \rho^* f, X \rangle$$

where $\mathcal{L}_{u_X}$ denotes the Lie derivative and $\langle\, , \rangle$ is the natural contraction between $V_G P$ and $V_G^* P$. It follows that Lemma 3.3 holds whenever the Higgs potential U is *gauge-invariant*, i.e., $\mathcal{L}_{u_X} U = 0$ for every $X: M \to V_G P$. ■

Finally let us show that the condition (iii′) of Section 2.2 holds.

Lemma 3.5. For every $p \in R^2$ there exists a quasiregular basis of $T_{\pi^2(p)}M$ for G^2 at p.

Proof. Let $p \in R^2$ and let (dx^λ) be an orthonormal basis of $T_x^* M$, with $x = \pi^2(p)$. We have the following dimension counting:

$$\dim(G^2)_{p,m-1} = \dim G \tag{3.27}$$

$$\dim(G^2)_{p,j} = \left[\frac{(m-j)(m-j+1)}{2} - 1\right](m \dim G + \dim V),$$
$$1 \le j \le m-2 \tag{3.28}$$

$$\dim(G^2)_p = \left[\frac{m(m+1)}{2} - 1\right](m \dim G + \dim V) \tag{3.29}$$

and

$$\dim(G^3)_p = \frac{m^3 + 3m^2 - 4m}{6}(m \dim G + \dim V) + \dim G \tag{3.30}$$

The proof of (3.27) goes as follows. By definition, $(G^2)_{p,m-1}$ = Ker $\sigma(\Phi)_{m-1}$, where

$$\sigma(\Phi)_{m-1}: \quad \vee_{2,m-1}T_x^*M \otimes \overline{L}_x \to \overset{m-1}{\wedge} T_x^*M \otimes (V_G^*P)_x \oplus \overset{m}{\wedge} T_x^*M \otimes E_x^*$$

is the restriction of $\sigma(\Phi)$ to $\vee_{2,m-1} T_x^*M \otimes \overline{L}_x$. Using (3.16) and (3.17), we find

$$\xi_{m-1}(dx^m \vee dx^m \otimes dx^\lambda) = 2g^{mm}dx^\lambda - 2g^{m\lambda}dx^m, \qquad 1 \le \lambda \le m$$

$$\zeta_{m-1}(dx^m \vee dx^m) = 2g^{mm} \otimes \omega$$

thereby showing that dim(Ker ξ_{m-1}) = 1 and dim(Ker ζ_{m-1}) = 0. Hence

$$\dim(G^2)_{p,m-1} = \dim(\mathrm{Ker}\ \xi_{m-1})\dim G + \dim(\mathrm{Ker}\ \zeta_{m-1})\dim V = \dim G$$

To prove (3.28), note that

$$\sigma(\Phi)_j: \quad \vee_{2,j}T_x^*M \otimes \overline{L}_x \to \overset{m-1}{\wedge} T_x^*M \otimes (V_G^*P)_x \oplus \overset{m}{\wedge} T_x^*M \otimes E_x^*, \qquad 1 \le j \le m-2$$

is surjective and hence the sequence

$$0 \to (G^2)_{p,j} \to \vee_{2,j}T_x^*M \otimes \overline{L}_x \xrightarrow{\sigma(\Phi)_j} \overset{m-1}{\wedge} T_x^*M \otimes (V_G^*P)_x \oplus \overset{m}{\wedge} T_x^*M \otimes E_x^* \to 0$$

is exact. The equality (3.29) is an immediate consequence of Lemma 3.1. As for (3.30), from Lemma 3.2 we find

$$\begin{aligned}
\dim(G^3)_p &= \binom{m+2}{3}(m \dim G + \dim V) - \dim[\mathrm{Im}\ \sigma_1(\Phi)] \\
&= \binom{m+2}{3}(m \dim G + \dim V) - \dim[\mathrm{Ker}\ \sigma(\Psi)] \\
&= \left[\binom{m+2}{3} - m\right](m \dim G + \dim V) + \dim G \\
&= \frac{m^3 + 3m^2 - 4m}{6}(m \dim G + \dim V) + \dim G
\end{aligned}$$

It follows that

$$\dim(G^2)_p + \sum_{j=1}^{m-1} \dim(G^2)_{p,j}$$

$$= \left[\frac{m(m+1)}{2} - 1\right](m \dim G + \dim V)$$

$$+ \sum_{j=1}^{m-2} \left[\frac{(m-j)(m-j+1)}{2} - 1\right](m \dim G + \dim V)$$

$$+ \dim G$$

and since

$$\sum_{j=1}^{m-2} \frac{(m-j)(m-j+1)}{2} = \frac{m^3 - m - 6}{6}$$

we find

$$\dim(G^2)_p + \sum_{j=1}^{m-1} \dim(G^2)_{p,j}$$

$$= \frac{m^3 + 3m^2 - 4m}{6}(m \dim G + \dim V) + \dim G = \dim(G^3)_p \quad \blacksquare$$

Summarizing the discussion of this section, we have shown that the Yang–Mills–Higgs pde (3.11) is involutive and hence formally integrable. Now assume that the principal bundle $P \to M$ and the metric g of M are analytic. Then R^2 in (3.11) is an analytic pde and Theorem 2.1 leads us directly to the following theorem.

Theorem 3.6. Let the principal bundle $P \to M$ and the metric g be analytic. For any $x \in M$ there is an analytic solution (actually many analytic solutions) (A, ϕ) of the Yang–Mills–Higgs pde's over a neighborhood of x.

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